

Journal of Functional Analysis **196**, 87–118 (2002)
doi:10.1006/jfan.2002.3988

Non-topological Multivortex Solutions to the Self-Dual Maxwell–Chern–Simons–Higgs Systems*

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Received August 29, 2001; revised January 9, 2002; accepted April 8, 2002

In this paper we construct non-topological multivortex solutions to the non-relativistic self-dual Maxwell–Chern–Simons–Higgs system in \mathbb{R}^2 which make the energy functional finite. Moreover, our proof of the existence of solutions reveals precise asymptotic behavior of solutions near spatial infinity. Using exactly the same method, we also establish the existence of non-topological multivortex solutions to the relativistic self-dual Maxwell–Chern–Simons–Higgs system. © 2002 Elsevier Science (USA)

Key Words: Maxwell–Chern–Simons theory; Non-topological solutions.

0. INTRODUCTION

This paper is concerned with the existence of non-topological solutions to both relativistic and non-relativistic Maxwell–Chern–Simons–Higgs models. We shall briefly review the history of studies of the Chern–Simons models. The non-relativistic self-dual Chern–Simons model was introduced (see also [6] for a general survey of related models) by Jakiw and Pi [10] in order to explain quantum Hall effect, anyonic superconductivity, etc. Interestingly enough, the associated self-duality equations can be reduced to the well-known integrable equation—the Liouville equation, and hence, are exactly

*This research is supported partially by Korea Research Foundation Grant (KRF-2000-015-0409-20000026), and KIAS-M97003.

soluble. Later, in [7] the model was generalized to include the Maxwell term in the Lagrangian density. To make the theory self-dual they found, using the supersymmetry argument, it is necessary to have an extra neutral scalar field in the Lagrangian. Unlike the pure Chern–Simons model case, the associated self-duality equations (or, the Bogomol’nyi equations) of our non-relativistic Maxwell–Chern–Simons–Higgs system are reduced to a complicated system of nonlinear elliptic partial differential equations, and it seems hopeless to find an exact solution of them.

Under the periodic boundary conditions Spruck and Yang [18] rigorously proved existence of solutions of this system, and later Tarantello [20] refined their results proving, in particular, the existence of solutions for wider range of parameters in the equations. For the self-duality equations in \mathbb{R}^2 with suitable boundary conditions near infinity to guarantee the finiteness of the energy functional (the non-topological boundary condition), however, the existence of multivortex solution has been an open problem. The difficulty in that problem was not surprising, since with a similar boundary condition near infinity even the much simpler version of the problem—the existence of a general *non-topological* multivortex solution of the *relativistic self-dual Chern–Simons* system, introduced in [9, 11]—has been open until very recently; we note that Spruck and Yang [16] constructed only a radially symmetric non-topological solution of this system, and further analysis for the radial solutions was carried out in [5]. We remark that for the topological solutions, the existence of solutions and a constructive approximation scheme are studied in [17, 22], respectively; for the periodic boundary conditions, the existence and multiplicity of solutions are studied in [1, 19], respectively. We also remark that for the topological multivortex solutions of the relativistic self-dual Maxwell–Chern–Simons system in \mathbb{R}^2 , modeled in [14], the existence, asymptotic decays and various limiting properties of solutions were established in [2], and for the periodic boundary condition similar analysis is done in [3, 15]. Recently, in [4], however, the authors of the current paper succeeded the construction of non-topological multivortex solutions of the general type in the relativistic self-dual Chern–Simons system.

In this paper we extend substantially the method developed in [4], and apply it to our more complicated system of self-duality equations. One of our main results is the existence of multivortex solutions of the general type of the *non-relativistic self-dual Maxwell–Chern–Simons* system. We also find very precise information on the asymptotic behavior of the solutions near infinity. Moreover we find that for the relativistic self-dual Maxwell–Chern–Simons system the associated self-duality equations in \mathbb{R}^2 with the non-topological boundary conditions have exactly the same structure as those of the non-relativistic self-dual Maxwell–Chern–Simons system after a simple nonlinear transform of equations. By applying exactly the same argument

we also establish the existence of general non-topological multivortex solutions for this system.

1. NON-RELATIVISTIC SELF-DUAL MAXWELL–CHERN–SIMONS SYSTEMS

The Lagrangian density for the non-relativistic self-dual Maxwell–Chern–Simons–Higgs system introduced in [7] is

$$\begin{aligned} \mathcal{L}(A, \psi, \mathcal{N}) = & -\frac{1}{4q^2} F_{\mu\nu} F^{\mu\nu} + \frac{\gamma}{4q^2} \varepsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} + i\bar{\psi} D_0 \psi - \frac{1}{2m} D_\mu \psi \overline{D^\mu \psi} \\ & - \frac{1}{2q^2} \partial_\mu \mathcal{N} \partial^\mu \mathcal{N} - |\psi|^2 \mathcal{N} + \frac{1}{2} \left(\frac{q}{2m} |\psi|^2 - \frac{\gamma}{q} \mathcal{N} \right)^2, \end{aligned}$$

where $D_A = D_\mu dx^\mu$, $D_\mu = \partial_\mu - iA_\mu$, $\mu = 0, 1, 2$, is the covariant derivative associated with the gauge field $A = (A_0, A_1, A_2)$, $F_A = -\frac{i}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mu = 0, 1, 2$, is the associated curvature and $\psi = \psi_1 + i\psi_2$ is the Higgs field and \mathcal{N} is a neutral scalar field, $\varepsilon^{\mu\nu\rho}$ is the totally skew-symmetric tensor with $\varepsilon^{012} = 1$, $q > 0$ is the charge of electron, $2\gamma/q^2 = \kappa > 0$ is the Chern–Simons coupling constant, and $m > 0$ is the mass of the Higgs particle. Here we are using the Minkowski metric $g_{\mu\nu} = \text{diag}(-1, 1, 1)$, and the summation convention for the repeated indices as usual.

The static energy functional of the model is

$$\begin{aligned} \mathcal{E}(A, \psi, \mathcal{N}) = & \int_{\mathbb{R}^2} \left\{ \frac{1}{2q^2} (F_{j0}^2 + F_{12}^2) + \frac{1}{m} |D_j \psi|^2 + \frac{1}{2q^2} |\nabla \mathcal{N}|^2 \right. \\ & \left. - |\psi|^2 \mathcal{N} + \frac{1}{2} \left(\frac{q}{2m} |\psi|^2 - \frac{\gamma}{q} \mathcal{N} \right)^2 \right\} dx. \end{aligned}$$

The self-duality equations coupled with the Gauss law constraint (the variational equation for \mathcal{L} with respect to A_0) for the static solutions (A, ψ, \mathcal{N}) are [7]

$$A_0 = -\mathcal{N}, \quad (D_1 + iD_2)\psi = 0 \quad \text{in } \mathbb{R}^2, \quad (1.1)$$

$$\partial_1 A_2 - \partial_2 A_1 + \frac{q^2}{4\gamma} |\psi|^2 - \gamma \mathcal{N} = 0 \quad \text{in } \mathbb{R}^2, \quad (1.2)$$

$$(\Delta - \gamma^2)\mathcal{N} + \frac{q^2}{2} \left(1 + \frac{\gamma}{2m} \right) |\psi|^2 = 0 \quad \text{in } \mathbb{R}^2, \quad (1.3)$$

$$A(x), \psi(x), \mathcal{N}(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (1.4)$$

Boundary condition (1.4) results from the requirement that the energy functional should be finite. Following Jaffe–Taubes [12], we can reduce the system (1.1)–(1.4) to an elliptic system of partial differential equations by introducing the new unknown function $u = \ln |\psi|^2$, with

$$\psi = \exp \left[\frac{u}{2} + i \sum_{j=1}^k n_j \arg(z - z_j) \right], \quad z = x_1 + ix_2 \in \mathbb{C}^1 = \mathbb{R}^2,$$

where $\{z_j\}_{j=1}^k$ and $\{n_j\}_{j=1}^k$ are the *prescribed* zeros of ψ and their multiplicities, respectively; the solutions (A, ψ, \mathcal{N}) of (1.1)–(1.4) are called *multivortex solutions* with the centers of vorticities on $\{z_j\}_{j=1}^k$. In this case system (1.1)–(1.4) reduces to

$$\Delta u = \frac{q^2}{2m} e^u - 2\gamma \mathcal{N} + 4\pi \sum_{j=1}^k n_j \delta(z - z_j) \quad \text{in } \mathbb{R}^2, \quad (1.5)$$

$$\Delta \mathcal{N} = \gamma^2 \mathcal{N} - \frac{q^2}{2} \left(1 + \frac{\gamma}{2m} \right) e^u \quad \text{in } \mathbb{R}^2, \quad (1.6)$$

$$u(x) \rightarrow -\infty, \quad \mathcal{N}(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (1.7)$$

The boundary condition (1.7) is called *non-topological*, since $\int_{\mathbb{R}^2} F_{12} dx$ is not an integer multiple of some fixed quantity under this boundary condition; for more detailed discussions including the physical meanings of this boundary condition see [7]. Our aim is to construct a solution (A, ψ, \mathcal{N}) to (1.1)–(1.4) (or, equivalently (1.5)–(1.7)) with prescribed $\{z_j\}_{j=1}^k$ and $\{n_j\}_{j=1}^k$ that makes the energy functional $\mathcal{E}(A, \psi, \mathcal{N})$ finite. In order to formulate our results we introduce the functions

$$\rho_{\varepsilon, a}(z) = \frac{8\gamma \varepsilon^{2N+2} |f(z)|^2}{q^2 (1 + \varepsilon^{2N+2} |F(z) + \frac{a}{\varepsilon^{N+1}}|^2)^2}, \quad \rho(r) = \frac{8(N+1)^2 r^{2N}}{(1 + r^{2N+2})^2}, \quad (1.8)$$

where $a = a_1 + ia_2$, $z = x_1 + ix_2$, $r = |z|$ and $f(z), F(z)$ are the complex functions

$$f(z) = (N+1) \prod_{j=1}^k (z - z_j)^{n_j}, \quad F(z) = \int_0^z f(\xi) d\xi, \\ N = \sum_{j=1}^k n_j, \quad n_j \geq 0 \quad (1.9)$$

and the function $w_0(|z|)$ is defined by

$$w_0(r) = \varphi_0(r) \left\{ \int_0^r \frac{\phi_{\tilde{f}}(s) - \phi_{\tilde{f}}(1)}{(1-s)^2} ds + \frac{\phi_{\tilde{f}}(1)r}{1-r} \right\} \quad (1.10)$$

with

$$\phi_{\tilde{f}}(r) = \left(\frac{1+r^{2N+2}}{1-r^{2N+2}} \right)^2 \frac{(1-r)^2}{r} \int_0^r \varphi_0(t) t \tilde{f}(t) dt, \quad (1.11)$$

$$\begin{aligned} \varphi_0(r) &= \frac{1-r^{2N+2}}{1+r^{2N+2}}, & \Psi_0(r) &= \left(\frac{1}{2\gamma^3} + \frac{1}{4\gamma^2 m} \right) \left(\frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 - \rho^2 \right), \\ \tilde{f}(r) &= -2\gamma \Psi_0, \end{aligned} \quad (1.12)$$

where $\phi_{\tilde{f}}(1)$ and $w(1)$ are defined as limits of $\phi_{\tilde{f}}(r)$ and $w(r)$ as $r \rightarrow 1$.

The following is our main result for the *non-relativistic* self-dual Chern–Simons–Higgs system.

THEOREM 1.1. *Let $\{z_j\}_{j=1}^k \subset \mathbb{C}^1$, $\{n_j\}_{j=1}^k \subset \mathbb{Z}_+$, $N = \sum_{j=1}^k n_j$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists a solution $(\psi_\varepsilon, A_\varepsilon, \mathcal{N}_\varepsilon)$ to problem (1.1)–(1.4) with the following properties:*

1. *The energy functional $\mathcal{E}(\psi_\varepsilon, A_\varepsilon, \mathcal{N}_\varepsilon)$ is finite, and the function ψ_ε has the zeros $\{z_j\}_{j=1}^k$ with multiplicities $\{n_j\}_{j=1}^k$.*
2. *The solutions $(\psi_\varepsilon, A_\varepsilon, \mathcal{N}_\varepsilon)$ could be represented by formulas*

$$\psi_\varepsilon(z) = e^{[\frac{\mu_\varepsilon}{2} + i \sum_{j=1}^k n_j \arg(z - z_j)]}, \quad (1.13)$$

$$u_\varepsilon(z) = \ln \rho_{\varepsilon, a_\varepsilon}(z) + \varepsilon^2 w_0(\varepsilon|z|) + u_\varepsilon(z), \quad (1.14)$$

$$\mathcal{N}_\varepsilon(z) = \frac{q^2}{2\gamma^2} \left(1 + \frac{\gamma}{2m} \right) e^{\mu_\varepsilon(z)} + \Psi_\varepsilon(z) + \varepsilon^4 \Psi_0(\varepsilon z), \quad (1.15)$$

where $a_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, and

$$w_0(\varepsilon|z|) = -\sigma \ln |z| + o(\ln |z|) \quad \text{as } |z| \rightarrow +\infty, \quad (1.16)$$

where

$$\sigma = \frac{8\pi N(N+2)(1 + \frac{\gamma}{2m})}{3(N+1)\gamma^2 \sin(\frac{\pi N}{N+1})},$$

$$\|\Psi_\varepsilon\|_{L^2(\mathbb{R}^2)} = o(\varepsilon^3), \quad \|u_\varepsilon / \ln(|z| + 1)\|_{\mathbb{C}^0(\mathbb{R}^2)} \leq o(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow +0. \quad (1.17)$$

3. *There exists a constant $\hat{C} = \hat{C}(\gamma, q, m)$ and a function $\beta(\varepsilon)$ such that solutions $(\psi_\varepsilon, A_\varepsilon, \mathcal{N}_\varepsilon)$ satisfy the decay estimates*

$$\ln |\psi_\varepsilon(z)|^2 = -(2N + 4 + \sigma\varepsilon^2 + o(\varepsilon^2)) \ln |z| + o(\ln |z|) \quad \text{as } |z| \rightarrow \infty, \quad (1.18)$$

$$|D_1\psi_\varepsilon(z)|^2 + |D_2\psi_\varepsilon(z)|^2 \leq \frac{\hat{C}}{|z|^{2N+4+\beta(\varepsilon)}} + o\left(\frac{1}{|z|^{2N+4+\beta(\varepsilon)}}\right) \quad \text{as } |z| \rightarrow \infty, \quad (1.19)$$

where $\beta(\varepsilon) > 0$ for all $\varepsilon \in (0, \varepsilon_0)$ and $\lim_{\varepsilon \rightarrow +0} \beta(\varepsilon)/\varepsilon^2 = \sigma$.

4. For any $z \in \mathbb{C}^1$ the following pointwise inequality holds:

$$\frac{q^2}{4\gamma m} |\psi_\varepsilon(z)|^2 \leq \mathcal{N}_\varepsilon(z). \quad (1.20)$$

Remark 1.1. In Theorem 1.1 relation (1.20) does not depend on our construction, and holds for *any* smooth solutions of (1.1)–(1.4).

Throughout this paper we identify $z = x_1 + ix_2 \in \mathbb{C}^1$ with $x = (x_1, x_2) \in \mathbb{R}^2$.

2. FUNCTIONAL SETTING OF THE PROBLEM

First we transform system (1.5)–(1.7) into a new one. Let us introduce a change of variables from (\mathcal{N}, u) into (S, u) defined by the equation

$$\mathcal{N} = \frac{q^2}{2\gamma^2} \left(1 + \frac{\gamma}{2m}\right) e^u + S. \quad (2.1)$$

Then Eq. (1.5) is transformed into

$$\Delta u = -\frac{q^2}{\gamma} e^u - 2\gamma S + 4\pi \sum_{j=1}^k n_j \delta(z - z_j), \quad (2.2)$$

and then, using (2.2), Eqs. (1.6) and (1.7) are transformed into

$$\begin{aligned} \Delta S &= \gamma^2 S - \frac{q^2}{2\gamma^2} \left(1 + \frac{\gamma}{2m}\right) |\nabla u|^2 e^u \\ &\quad + \frac{q^4}{2\gamma^3} \left(1 + \frac{\gamma}{2m}\right) e^{2u} + \frac{q^2}{\gamma} \left(1 + \frac{\gamma}{2m}\right) S e^u, \end{aligned} \quad (2.3)$$

$$u(x) \rightarrow -\infty, \quad S(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad (2.4)$$

respectively.

Next we change variables from (S, u) to (S, v) by

$$u = v + \ln \rho_{\varepsilon, a},$$

where $\rho_{\varepsilon,a}$ is determined by formulas (1.8) and (1.9). It is well known that for any $\varepsilon > 0$ and $a \in \mathbb{C}^1$, the function $\Phi(z) = \ln \rho_{\varepsilon,a}(z)$ is a solution of the Liouville equation

$$\Delta \Phi = -\frac{q^2}{\gamma} e^\Phi + 4\pi \sum_{j=1}^k n_j \delta(z - z_j) \quad \text{in } \mathbb{R}^2. \quad (2.5)$$

Then, (2.2) and (2.3) and reduced to

$$\Delta v = -\hat{a} \rho_{\varepsilon,a} e^v + \hat{a} \rho_{\varepsilon,a} - 2\gamma S, \quad (2.6)$$

$$\Delta S = \gamma^2 S - \hat{b} |\nabla(v + \ln \rho_{\varepsilon,a})|^2 e^v \rho_{\varepsilon,a} + \hat{a} \hat{b} e^{2v} \rho_{\varepsilon,a}^2 + 2\hat{b} \gamma e^v \rho_{\varepsilon,a} S, \quad (2.7)$$

where we set $\hat{a} = \frac{q^2}{\gamma}$, $\hat{b} = \frac{q^2}{2\gamma^2} (1 + \frac{\gamma}{2m})$. On the other hand, the functions

$$\tilde{v}(x) = v\left(\frac{x}{\varepsilon}\right), \quad \tilde{S}(x) = \frac{1}{\varepsilon^2} S\left(\frac{x}{\varepsilon}\right)$$

satisfy the equations

$$\Delta \tilde{v} = -\hat{a} e^{\tilde{v}} g_{\varepsilon,a} + \hat{a} g_{\varepsilon,a} - 2\gamma \tilde{S},$$

$$\varepsilon^2 \Delta \tilde{S} = \gamma^2 \tilde{S} - \varepsilon^2 \hat{b} |\nabla(\tilde{v} + \ln g_{\varepsilon,a})|^2 e^{\tilde{v}} g_{\varepsilon,a} + \varepsilon^2 \hat{a} \hat{b} e^{2\tilde{v}} g_{\varepsilon,a}^2 + 2\hat{b} \gamma \varepsilon^2 e^{\tilde{v}} g_{\varepsilon,a} \tilde{S},$$

where

$$g_{\varepsilon,a}(x) = \frac{1}{\varepsilon^2} \rho_{\varepsilon,a}\left(\frac{x}{\varepsilon}\right).$$

Recall that

$$\rho(r) = \frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2}, \quad \Psi_0(r) = \frac{\hat{b}}{\gamma^2 \hat{a}} \left(\frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 - \rho^2 \right).$$

In order to construct a good approximate solution to our problem we consider the ordinary differential equation

$$L_1 w_0 = \frac{1}{r} \frac{d}{dr} \left(r \frac{dw_0}{dr} \right) + \rho w_0 = -2\gamma \Psi_0 \quad \text{in } \mathbb{R}_+. \quad (2.8)$$

LEMMA 2.1. *The function $w_0(r)$ defined by (1.10)–(1.12) solves Eq. (2.8) and satisfies the pointwise estimate*

$$|w_0(r)| \leq C(\ln^+ r + 1), \quad \forall r > 0 \quad (2.9)$$

holds true. Moreover, we have the asymptotic formula

$$w_0(r) = -\frac{8\pi N(N+2)(1+\frac{\gamma}{2m})}{3(N+1)\gamma^2 \sin(\frac{\pi N}{N+1})} \ln r + o(\ln r) \quad \text{as } r \rightarrow \infty. \quad (2.10)$$

The proof of this lemma is given in Section 5. Now let us make the change

$$\tilde{v} = \varepsilon^2 u + \varepsilon^2 w_0, \quad \tilde{S} = \varepsilon^2 \Psi + \varepsilon^2 \Psi_0.$$

By elementary calculation we obtain the equations

$$\Delta u + \hat{a}g_{\varepsilon,a} \frac{(e^{\varepsilon^2(u+w_0)} - 1)}{\varepsilon^2} + 2\gamma\Psi + \Delta w_0 + 2\gamma\Psi_0 = 0, \quad (2.11)$$

$$\begin{aligned} & \varepsilon^2 \Delta \Psi - \gamma^2 \Psi + \hat{b} |\nabla(\varepsilon^2 u + \varepsilon^2 w_0 + \ln g_{\varepsilon,a})|^2 e^{\varepsilon^2 u + \varepsilon^2 w_0} g_{\varepsilon,a} \\ & - \hat{a} \hat{b} e^{2(\varepsilon^2 u + \varepsilon^2 w_0)} g_{\varepsilon,a}^2 - 2\hat{b} \gamma e^{\varepsilon^2 u + \varepsilon^2 w_0} g_{\varepsilon,a} (\varepsilon^2 \Psi + \varepsilon^2 \Psi_0) \\ & - \gamma^2 \Psi_0 + \varepsilon^2 \Delta \Psi_0 = 0. \end{aligned} \quad (2.12)$$

In order to make a next step in the transformation of (2.11) and (2.12) we should solve some boundary-value problem for an elliptic operator in weighted Sobolev spaces. First, let us introduce these function spaces:

$$X = \left\{ u(x) \in L_{\text{loc}}^2(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} (1 + |x|^{2+\frac{1}{4}}) u^2 dx < \infty \right\},$$

equipped with the inner product $(u, v)_X = \int_{\mathbb{R}^2} (1 + |x|^{2+\frac{1}{4}}) uv dx$, and

$$Y = \left\{ u \in W_{\text{loc}}^{2,2}(\mathbb{R}^2) \mid \|\Delta u\|_X^2 + \left\| \frac{u}{1 + |x|^{1+\frac{1}{8}}} \right\|_{L^2(\mathbb{R}^2)}^2 < \infty \right\}$$

equipped with the inner product

$$(u, v)_Y = (\Delta u, \Delta v)_X + \int_{\mathbb{R}^2} \frac{uv}{1 + |x|^{2+\frac{1}{4}}} dx.$$

These spaces are equipped with the natural Banach space norms

$$\|u\|_X = \sqrt{(u, u)_X}, \quad \|u\|_Y = \sqrt{(u, u)_Y},$$

respectively. Thanks to the inequality

$$\int_{\mathbb{R}^2} |u| dx \leq \left(\int_{\mathbb{R}^2} \frac{1}{1 + |x|^{2+\frac{1}{4}}} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} (1 + |x|^{2+\frac{1}{4}}) u^2 dx \right)^{\frac{1}{2}}$$

there is a continuous imbedding

$$X \hookrightarrow L^1(\mathbb{R}^2). \quad (2.13)$$

Also, by the local regularity of the Laplace operator (see e.g. [8]) we have

$$Y \subset C_{\text{loc}}^0(\mathbb{R}^2).$$

Denote $\ln^+ r = \max\{0, \ln r\}$. For functions from the space Y we have the following growth estimate near infinity:

LEMMA 2.2 Chae and Yu Imanuvilov [4]. *There exists $C_1 > 0$ such that for all $v \in Y$,*

$$|v(x)| \leq C_1 \|v\|_Y (\ln^+ |x| + 1), \quad \forall x \in \mathbb{R}^2. \quad (2.14)$$

Now we consider the following boundary-value problem for given $h \in X$:

$$\varepsilon_\varepsilon R = \varepsilon \Delta R - \gamma^2 R = h \quad \text{in } \mathbb{R}^2, \quad R(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (2.15)$$

LEMMA 2.3. *Let $\gamma > \gamma_0 > 0, \varepsilon \in (0, 1)$ be given. Then for any $h \in X$,*

$$\varepsilon^2 \sum_{i,j=1}^2 \|\partial_i \partial_j R\|_{L^2(\mathbb{R}^2)} + \varepsilon \gamma \|\nabla R\|_X + \gamma^2 \|R\|_X \leq C \|h\|_X, \quad (2.16)$$

where C is independent of γ, ε .

Proof. It is well known that Eq. (2.15), with zero boundary conditions on infinity, has the unique solution $R \in W^{2,2}(\mathbb{R}^2)$ which satisfies the estimates

$$\varepsilon^2 \|R\|_{W^{2,2}(\mathbb{R}^2)} \leq C(\gamma) \|h\|_X, \quad \gamma^2 \|R\|_{L^2(\mathbb{R}^2)} \leq C \|h\|_{L^2(\mathbb{R}^2)}, \quad (2.17)$$

where the constant C in the second inequality does not depend on γ, ε .

Now it is convenient for us, instead of (2.15), to consider a sequence of an auxiliary problems

$$\varepsilon \Delta R_k - \gamma^2 R_k = h \quad \text{in } B_k = \{x \mid |x| \leq k\}, \quad R_k|_{\partial B_k} = 0, \quad (2.18)$$

where $k \in \mathbb{Z}_+$. Obviously, the following estimates similar to (2.17) holds true

$$\varepsilon^2 \|R_k\|_{W^{2,2}(B_k)} \leq C(\gamma) \|h\|_X, \quad \gamma^2 \|R_k\|_{L^2(B_k)} \leq C \|h\|_{L^2(B_k)}, \quad (2.19)$$

with constants C independent of k .

Taking scalar product of (2.18) with $\eta_1 R_k$, where $\eta_1(x) = (1 + |x|^2)$, we obtain

$$\int_{B_k} \left(\varepsilon |\nabla R_k|^2 \eta_1 + \gamma^2 \eta_1 R_k^2 - \frac{\varepsilon}{2} \Delta \eta_1 R_k^2 \right) dx = - \int_{B_k} \eta_1 h R_k dx. \quad (2.20)$$

Due to the identity $\Delta \eta_1 = 4$, we estimate, using (2.19),

$$\begin{aligned} \gamma^2 \int_{B_k} \eta_1 R_k^2 dx &\leq 2 \int_{B_k} R_k^2 dx + \left(\int_{B_k} \eta_1 R_k^2 dx \right)^{\frac{1}{2}} \left(\int_{B_k} \eta_1 h^2 dx \right)^{\frac{1}{2}} \\ &\leq \frac{2C}{\gamma^2} \|h\|_{L^2(\mathbb{R}^2)}^2 + \frac{\gamma^2}{2} \int_{B_k} \eta_1 R_k^2 dx + \frac{C}{\gamma^2} \int_{\mathbb{R}^2} \eta_1 h^2 dx. \end{aligned}$$

Thus, we have

$$\gamma^4 \int_{B_k} (1 + |x|^2) R_k^2 dx \leq C \|h\|_X^2. \quad (2.21)$$

Next, we multiply (2.18) by $\gamma^2 \eta_2 R_k$ with $\eta_2(x) = (1 + |x|^2)^{\frac{1}{4}}$, scalarly in $L^2(B_k)$. Similarly, we obtain

$$\int_{B_k} \left(\varepsilon \gamma^2 |\nabla R_k|^2 \eta_2 + \gamma^4 \eta_2 R_k^2 - \frac{\gamma^2}{2} \Delta \eta_2 R_k^2 \right) dx = - \int_{B_k} \gamma^2 \eta_2 h R_k dx.$$

Since $\Delta \eta_2 \leq C(1 + |x|^{\frac{1}{4}})$, using estimate (2.21), we deduce that

$$\begin{aligned} \int_{B_k} (\varepsilon \gamma^2 \eta_2 |\nabla R_k|^2 + \gamma^4 \eta_2 R_k^2) dx &\leq C \gamma^2 \int_{B_k} (1 + |x|^{\frac{1}{4}}) R_k^2 dx \\ &\quad + \gamma^2 \left(\int_{B_k} \eta_2 R_k^2 dx \right)^{\frac{1}{2}} \left(\int_{B_k} \eta_2 h^2 dx \right)^{\frac{1}{2}} \\ &\leq C \|h\|_X^2 + \frac{\gamma^4}{2} \int_{B_k} \eta_2 R_k^2 dx \\ &\quad + C \int_{B_k} \eta_2 h^2 dx. \end{aligned} \quad (2.22)$$

Then, taking, if it is necessary, a subsequence, we can assume

$$R_k \rightarrow \tilde{R} \quad \text{in } W^{1,2}(\mathbb{R}^2) \text{ as } k \rightarrow +\infty$$

and by (2.21) and (2.22)

$$\varepsilon^2 \gamma \|\nabla \tilde{R}\|_X + \gamma \|\tilde{R}\|_X \leq C \|h\|_X. \quad (2.23)$$

Obviously \tilde{R} is a solution to (2.16), and due to the uniqueness theorem for this problem, we have $\tilde{R} \equiv R$. On the other hand, by the Calderon–Zygmund inequality (see e.g. [8])

$$\begin{aligned} \varepsilon^2 \sum_{i,j=1}^2 \|\partial_i \partial_j R\|_{L^2(\mathbb{R}^2)} &\leq C \|\Delta R\|_{L^2(\mathbb{R}^2)} \\ &\leq C(\gamma^2 \|R\|_{L^2(\mathbb{R}^2)} + \|h\|_{L^2(\mathbb{R}^2)}) \\ &\leq C \|h\|_{L^2(\mathbb{R}^2)} \leq C \|h\|_X, \end{aligned} \quad (2.24)$$

where we used (2.17), and the constant C in the last inequality does not depend on γ . Estimates (2.23) and (2.24) imply (2.16). ■

By Lemma 2.2 there exists the operator $E_\varepsilon^{-1} : X \rightarrow X$ and the norm $\|E_\varepsilon^{-1}\|_{\mathcal{L}(X,X)}$ is¹ uniformly bounded for $\varepsilon \in [0, 1]$. Let us introduce the mapping $G_1(\cdot, \cdot, \cdot, \cdot) : Y \times \mathbb{R}^3 \times X \mapsto X$ as follows. For any $(u, a, \varepsilon, f) \in Y \times \mathbb{R}^3 \times X$ we set $G_1(u, \varepsilon, a, f) = \Phi$, where $\Phi(x)$ is a solution to the problem

$$\begin{aligned} K(u, a, \varepsilon)\Phi &= \varepsilon^2 \Delta \Phi - \gamma^2 \Phi - 2\varepsilon^2 \hat{b} \gamma e^{\varepsilon^2 u + \varepsilon^2 w_0} g_{\varepsilon, a} \Phi = f, \\ \Phi(x) &\rightarrow 0 \text{ as } |x| \rightarrow \infty. \end{aligned} \quad (2.25)$$

We set

$$\Omega_\delta = \{(u, a, \varepsilon) \in Y \times \mathbb{R}^3 \mid |\varepsilon| \leq \delta, |a| \leq \delta, \|u\|_Y \leq \delta\}.$$

PROPOSITION 2.1. *There exists $\delta > 0$ such that the mapping G_1 is well defined on $\Omega_\delta \times X$.*

Proof. Note that

$$|g_{\varepsilon, a}(x)| = O(|x|^{-4N-4}) \quad \text{as } |x| \rightarrow \infty.$$

On the other hand, by (2.10) and (2.14)

$$e^{2(\varepsilon^2 u(x) + \varepsilon^2 w_0(x))} \leq e^{2\varepsilon^2 C(\|u\|_Y + 1)\ln(|x| + 2)} \leq (2 + |x|)^{2\varepsilon^2 C(\|u\|_Y + 1)}.$$

Thus, for any $\varepsilon_1 > 0$ there exists $\delta(\varepsilon_1) > 0$ such that

$$|2\varepsilon^2 \hat{b} \gamma e^{\varepsilon^2 u + \varepsilon^2 w_0} g_{\varepsilon, a}| \leq \frac{\varepsilon_1}{1 + |x|^{4N+3}}, \quad \forall x \in \mathbb{R}^2, \quad (u, a, \varepsilon) \in \Omega_\delta. \quad (2.26)$$

¹By $\mathcal{L}(B_1, B_2)$ we denote the space of linear operators from the Banach space B_1 to the Banach space B_2 .

Applying the operator E_ε^{-1} to Eq. (2.25), we have

$$(I - E_\varepsilon^{-1}(2\varepsilon^2 \hat{b} \gamma e^{\varepsilon^2 u + \varepsilon^2 w_0} g_{\varepsilon, a})) \Phi = E_\varepsilon^{-1} f.$$

Thanks to (2.26) one could take $\delta > 0$ such that

$$\|E_\varepsilon^{-1}(2\varepsilon^2 \hat{b} \gamma e^{\varepsilon^2 u + \varepsilon^2 w_0} g_{\varepsilon, a})\|_{\mathcal{L}(X, X)} \leq \frac{1}{2}, \quad \forall (u, a, \varepsilon) \in \Omega_\delta.$$

On the other hand, (2.16) and well-known arguments (see e.g. [23, p. 32]) imply that there exists a unique solution $\Phi \in X$ to Eq. (2.25). ■

We set

$$\begin{aligned} G_2(u, a, \varepsilon) = & -\hat{b} |\nabla(\varepsilon^2 u + \varepsilon^2 w_0 + \ln g_{\varepsilon, a})|^2 e^{\varepsilon^2 u + \varepsilon^2 w_0} g_{\varepsilon, a} + \hat{a} \hat{b} e^{2(\varepsilon^2 u + \varepsilon^2 w_0)} g_{\varepsilon, a}^2 \\ & + 2\varepsilon^2 \hat{b} \gamma e^{\varepsilon^2 u + \varepsilon^2 w_0} g_{\varepsilon, a} \Psi_0 + \gamma^2 \Psi_0 - \varepsilon^2 \Delta \Psi_0. \end{aligned}$$

Solving Eq. (2.12) with respect to Ψ , we obtain

$$\Psi = G_1(u, a, \varepsilon, G_2(u, a, \varepsilon)).$$

Using this formula, one can rewrite (2.11) as $P(u, a, \varepsilon) = 0$, where

$$\begin{aligned} P(u, a, \varepsilon) = & \Delta u + \hat{a} g_{\varepsilon, a} \frac{e^{\varepsilon^2 u + \varepsilon^2 w_0} - 1}{\varepsilon^2} \\ & + 2\gamma G_1(u, a, \varepsilon, G_2(u, a, \varepsilon)) + \Delta w_0 + 2\gamma \Psi_0. \end{aligned} \quad (2.27)$$

Let $\varepsilon \mapsto (u_\varepsilon, a_\varepsilon)$ be an implicit function satisfying

$$P(u_\varepsilon, a_\varepsilon, \varepsilon) = 0,$$

then the pair $(u_\varepsilon, \mathcal{N}_\varepsilon)$, given by formulas

$$u_\varepsilon(x) = \ln \rho_{\varepsilon, a_\varepsilon}(x) + \varepsilon^2 u_\varepsilon(\varepsilon x) + \varepsilon^2 w_0(\varepsilon x), \quad (2.28)$$

$$\mathcal{N}_\varepsilon(x) = \frac{q^2}{2\gamma^2} \left(1 + \frac{\gamma}{2m}\right) e^{u_\varepsilon(x)} + \varepsilon^4 \Psi(\varepsilon x) + \varepsilon^4 \Psi_0(\varepsilon x) \quad (2.29)$$

with

$$\Psi(x) = G_1(u_\varepsilon, a_\varepsilon, \varepsilon, G_2(u_\varepsilon, a_\varepsilon, \varepsilon)) \quad (2.30)$$

is the solution to Eqs. (1.5) and (1.6). Of course one should check that this solution satisfies the boundary condition (1.7).

3. PROOF OF THE MAIN THEOREM

The mapping P constructed in the previous section is well defined for all $(u, a, \varepsilon) \in \Omega_\delta$ with $\varepsilon \neq 0$ where the parameter $\delta > 0$ is sufficiently small. For $\varepsilon = 0$ we set formally

$$P(u, a, 0) = \Delta u + \hat{a}g_{a,0}(u + w_0) + 2\gamma G_1(u, a, 0, G_2(u, a, 0)) + \Delta w_0 + 2\gamma \Psi_0.$$

Hence

$$P(0, 0, 0) = 0.$$

Our goal in this section is to obtain the parameterized solutions $(u_\varepsilon, a_\varepsilon)$ to equation

$$P(u_\varepsilon, a_\varepsilon, \varepsilon) = 0 \tag{3.1}$$

for all $\varepsilon \in (-\delta, \delta)$, where δ is a sufficiently small positive number. In order to prove this result we shall use the Implicit Function Theorem [23]. First let us establish some regularity properties of the mapping P at the point 0. We set

$$\varphi_+(r, \theta) = \frac{r^{N+1} \cos(N+1)\theta}{1 + r^{2N+2}}, \quad \varphi_-(r, \theta) = \frac{r^{N+1} \sin(N+1)\theta}{1 + r^{2N+2}}.$$

We have

PROPOSITION 3.1. *There exists $\delta_1 > 0$ such that the function $G_2 \in C \times (\Omega_{\delta_1}, X)$ and its partial derivatives $\frac{\partial G_2}{\partial a}, \frac{\partial G_2}{\partial u}$ are continuous on Ω_{δ_1} . Moreover,*

$$\frac{\partial G_2}{\partial u}(0)[\cdot] = 0$$

and

$$\begin{aligned} \frac{\partial G_2}{\partial a}(0)[b_1, b_2] &= 8 \frac{\hat{b}}{\hat{a}} \left(\frac{d\rho}{dr} \frac{\partial \varphi_+}{\partial r} b_1 + \frac{d\rho}{dr} \frac{\partial \varphi_-}{\partial r} b_2 \right) \\ &\quad + 4 \frac{\hat{b}}{\hat{a}} \frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 (\varphi_+ b_1 + \varphi_- b_2) \\ &\quad - 8 \frac{\hat{b}}{\hat{a}} \rho^2 (\varphi_+ b_1 + \varphi_- b_2). \end{aligned} \tag{3.2}$$

The proof of this proposition is based on Lemma 2.2, and uses only standard arguments. Thus we skip it. Now we establish some regularity properties of the mapping G_1 .

PROPOSITION 3.2. *There exists $\delta_2 > 0$ such that $G_1 \in C^1(\Omega_{\delta_2} \times X, X)$ and*

$$\frac{\partial G_1}{\partial u}(0)[h] = 0, \quad \frac{\partial G_1}{\partial a}(0)[b_1, b_2] = 0, \quad \frac{\partial G_1}{\partial f}(0)[\xi] = -\frac{\xi}{\gamma^2}. \quad (3.3)$$

Proof. We assume that $0 < \delta_2 < \delta$, where δ is the corresponding parameter from Proposition 2.1. We note that for fixed (u, a, ε) the mapping $G_1(u, a, \varepsilon, f)$ is the linear continuous mapping with respect to the variable f . Moreover this linear mapping is differentiable and formula (2.25) implies

$$\frac{\partial G_1}{\partial f}(u, a, \varepsilon, f)[\xi] = K^{-1}(u, a, \varepsilon)\xi,$$

where the operator $K(u, a, \varepsilon)$ is given by (2.25). In particular, for $(u, a, \varepsilon) = (0, 0, 0)$ we have

$$K^{-1}(0, 0, 0) = -\frac{1}{\gamma^2}.$$

Now let us show that

$$\|K^{-1}(u, a, \varepsilon) - K^{-1}(u_i, a_i, \varepsilon_i)\|_{\mathcal{L}(X; X)} \rightarrow 0 \quad \text{as } (u_i, a_i, \varepsilon_i) \rightarrow (u, a, \varepsilon) \text{ in } Y \times \mathbb{R}^3.$$

This follows, in turn, from the inequality

$$|e^{\varepsilon^2 u_i(x) + \varepsilon^2 w_0(x)} g_{\varepsilon_i, a_i} - e^{\varepsilon^2 u(x) + \varepsilon^2 w_0(x)} g_{\varepsilon, a}| \leq \frac{\hat{\varepsilon}_i}{1 + |x|^{4N+3}}, \quad \forall x \in \mathbb{R}^2,$$

where $\hat{\varepsilon}_i \rightarrow 0$ as $i \rightarrow \infty$. Now we claim

$$\frac{\partial G_1}{\partial u}(u, a, \varepsilon, f)[h] = K^{-1}(u, a, \varepsilon)[-2\varepsilon^4 \gamma \hat{b} g_{\varepsilon, a} e^{\varepsilon^2 u + \varepsilon^2 w_0} G_1(u, a, \varepsilon, f)h]. \quad (3.4)$$

In order to prove (3.4) it suffices to show that the function $\Psi(h) = G_1 \times (u + h, a, \varepsilon, f) - G_1(u, a, \varepsilon, f) - \frac{\partial G_1}{\partial u}(u, a, \varepsilon, f)[h]$ satisfies

$$\|\Psi(h)\|_X = o(\|h\|_Y) \quad \text{as } \|h\|_Y \rightarrow 0. \quad (3.5)$$

Note that

$$2\varepsilon^2 \hat{b} \gamma^2 e^{\varepsilon^2 w_0 + \varepsilon^2(u+h)} g_{\varepsilon, a} = 2\varepsilon^2 \hat{b} \gamma^2 g_{\varepsilon, a} e^{\varepsilon^2 u + \varepsilon^2 w_0} (1 + \varepsilon^2 h) + R(\varepsilon, a, x),$$

where $R(\varepsilon, a, x) = 2\varepsilon^2 \hat{b} \gamma^2 g_{\varepsilon, a} \int_0^1 (1-t) e^4 e^{\varepsilon^2 w_0(x) + \varepsilon^2(u+th)(x)} h^2 dt$. Taking the parameter $\delta_2 > 0$ sufficiently small, thanks to (2.10) and (2.14) for all

$(u, a, \varepsilon) \in \Omega_{\delta_2}$ we have

$$|R(\varepsilon, a, x)(x)| \leq \frac{o(\|h\|_Y)}{1 + |x|^{4N+3}} \quad \text{as } \|h\|_Y \rightarrow 0.$$

Therefore

$$\begin{aligned} K(u, a, \varepsilon)\Psi(h) &= R(\varepsilon, a, x)G_1(u + h, a, \varepsilon, f) \\ &\quad + 2\varepsilon^4 \hat{b}\gamma^2 g_{\varepsilon, a} e^{\varepsilon^2 u + \varepsilon^2 w_0} (G_1(u + h, a, \varepsilon, f) - G_1(u, a, \varepsilon, f))h \\ &\quad \text{in } \mathbb{R}^2, \quad \Psi(h)(x) \rightarrow 0 \quad \text{as } |x| \rightarrow 0. \end{aligned}$$

Thus (3.5) follows from Proposition 2.1. Obviously, the mapping

$$(u, a, \varepsilon, f) \mapsto \frac{\partial G_1}{\partial u}(u, a, \varepsilon, f)$$

from $\Omega_\delta \times X$ into $\mathcal{L}(Y, X)$ is continuous. Similarly, one can prove the formula

$$\frac{\partial G_1}{\partial a}(u, a, \varepsilon, f)[b_1, b_2] = K^{-1}(u, a, \varepsilon) \left[-2\hat{b}\gamma\varepsilon^2 e^{\varepsilon^2 u + \varepsilon^2 w_0} \left(\frac{\partial g_{\varepsilon, a}}{\partial a_1} b_1 + \frac{\partial g_{\varepsilon, a}}{\partial a_2} b_2 \right) \right]$$

and the continuity of the mapping

$$(u, a, \varepsilon, f) \mapsto \frac{\partial G_1}{\partial a}$$

from $\Omega_\delta \times X$ to $\mathcal{L}(\mathbb{R}^2, X)$. The proof of the proposition is complete. \blacksquare

Now we prove the differentiability of the mapping P .

PROPOSITION 3.3. *There exists $\delta > 0$ such that the function $P \in C(\Omega_\delta, X)$ and its partial derivatives $\frac{\partial P}{\partial a}, \frac{\partial P}{\partial u}$ are continuous on Ω_δ .*

$$\begin{aligned} \mathbf{A}(h, b_1, b_2) &= \frac{\partial P}{\partial u}(0)[h] + \frac{\partial P}{\partial a}(0)[b_1, b_2] = Lh - 4\rho w_0(\varphi_+ b_1 + \varphi_- b_2) \\ &\quad - \frac{16\hat{b}}{\hat{a}\gamma} \frac{d\rho}{dr} \left(\frac{\partial \varphi_+}{\partial r} b_1 + \frac{\partial \varphi_-}{\partial r} b_2 \right) \\ &\quad - \left(\frac{8\hat{b}}{\hat{a}\gamma} \frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 - \frac{16\hat{b}}{\hat{a}\gamma} \rho^2 \right) (\varphi_+ b_1 + \varphi_- b_2), \end{aligned} \tag{3.6}$$

where $Lh = \Delta h + \rho h$.

Proof. Using Taylor's formula, we can rewrite formula (2.27) as follows:

$$P(u, a, \varepsilon) = \Delta u + \hat{a}g_{\varepsilon, a} \left[(u + w_0) + \int_0^1 (1-t)(u + w_0)^2 \varepsilon^2 e^{\varepsilon^2(u+w_0)t} dt \right] \\ + 2\gamma G_1(u, a, \varepsilon, G_2(u, a, \varepsilon)) + \Delta w_0 + 2\gamma \Psi_0.$$

By Propositions 3.1 and 3.2 the mapping $G_1(u, a, \varepsilon, G_2(u, a, \varepsilon))$ is continuous on Ω_δ . The definition of the spaces X and Y imply $\Delta \in \mathcal{L}(Y, X)$. By Lemmata 2.1 and 2.2 the mapping

$$G_3(u, a, \varepsilon) = \hat{a}g_{\varepsilon, a} \left((u + w_0) + \int_0^1 (1-t)(u + w_0)^2 \varepsilon^2 e^{\varepsilon^2(u+w_0)t} dt \right)$$

is continuous, with its partial derivatives with respect to u, a on Ω_δ , provided we choose $\delta > 0$ sufficiently small. By (2.8) and the formula

$$G_1(0, G_2(0)) = 0$$

we have $P(0) = 0$. Therefore the mapping P is continuous on Ω_δ . By the chain-rule there exists continuous partial derivatives of the mapping $G_1 \times (u, a, \varepsilon, G_2(u, a, \varepsilon))$ with respect to u and a . Therefore there exists continuous partial derivatives of the mapping P with respect to variables u and a . Short computations and Proposition 3.1 give formula (3.6). ■

Now we would like to study the properties of the operator \mathbf{A} .

PROPOSITION 3.4. *The following inequality holds true.*

$$\tilde{C}_\pm := \int_{\mathbb{R}^2} \left[\left\{ 4\rho w_0 + \frac{8\hat{b}}{\hat{a}\gamma} \left(\frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 - 2\rho^2 \right) \right\} \varphi_\pm^2 + \frac{16\hat{b}}{\hat{a}\gamma} \frac{d\rho}{dr} \frac{\partial \varphi_\pm}{\partial r} \varphi_\pm \right] dx < 0.$$

This proposition is proved in Section 5.

LEMMA 3.1. *The operator $\mathbf{A} : Y \times \mathbb{R}^2 \rightarrow X$ is epimorphic.*

Proof. For given $f \in X$, we have to show that there exist $u \in Y, a_1, a_2 \in \mathbb{R}$ such that $\mathbf{A}(u, a_1, a_2) = f$. Let us define $C_\pm = \int_{\mathbb{R}^2} f \varphi_\pm dx$, and let \tilde{C}_\pm be the non-zero constants defined in Proposition 3.4. Hence, the function, \tilde{f} introduced below, is well defined.

$$\begin{aligned} \tilde{f} = & f - \frac{C_+}{\tilde{C}_+} \left[4\rho w_0 \varphi_+ + \frac{8\hat{b}}{\hat{a}\gamma} \left(\frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 - 2\rho^2 \right) \varphi_+ + \frac{16\hat{b}}{\hat{a}\gamma} \frac{d\rho}{dr} \frac{\partial \varphi_+}{\partial r} \right] \\ & - \frac{C_-}{\tilde{C}_-} \left[4\rho w_0 \varphi_- + \frac{8\hat{b}}{\hat{a}\gamma} \left(\frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 - 2\rho^2 \right) \varphi_- + \frac{16\hat{b}}{\hat{a}\gamma} \frac{d\rho}{dr} \frac{\partial \varphi_-}{\partial r} \right]. \end{aligned}$$

Then, from $\int_0^{2\pi} \sin(N+1)\theta \cos(N+1)\theta d\theta = 0$ we obtain

$$\int_{\mathbb{R}^2} \tilde{f} \varphi_{\pm} dx = 0.$$

On the other hand, in [4] we proved

$$\text{Im } L = \left\{ f \in X \mid \int_{\mathbb{R}^2} f \varphi_{\pm} dx = 0 \right\}.$$

Thus, there exists $u \in Y$ such that $Lu = \tilde{f}$, and we have

$$\mathbf{A} \left(u, -\frac{C_+}{\tilde{C}_+}, -\frac{C_-}{\tilde{C}_-} \right) = f.$$

This completes the proof of the proposition. ■

Proof of Theorem 1.1. We consider the mapping $P(u, a, \varepsilon)$ in the domain $\hat{\Omega} = \Omega_\delta$ with sufficiently small $\delta > 0$. Thanks to Proposition 3.3 the mapping P and its partial derivatives $\frac{\partial P}{\partial u}, \frac{\partial P}{\partial a}$ are continuous on Ω_δ . By Lemma 3.1 the operator $P'_{(u,a)}(0, 0, 0) = \mathbf{A} : Y \times \mathbb{R}^2 \rightarrow X$ is surjective. Thus by the generalized implicit function theorem (see e.g. [21, p. 37]) there exist $\varepsilon_0 > 0$ and a continuous function $\varepsilon \mapsto (u_\varepsilon, a_\varepsilon) = v_\varepsilon$ from $(0, \varepsilon_0)$ into a neighborhood of 0 in $Y \times \mathbb{R}^2$ such that

$$P(u_\varepsilon, a_\varepsilon, \varepsilon) = 0, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Now we check that the solution $(u_\varepsilon, \mathcal{N}_\varepsilon)$ recovered from formulas (2.28)–(2.30) with $\Psi_\varepsilon = \varepsilon^4 \Psi(\varepsilon x)$, is really non-topological. From the explicit formula for the function $\rho_{\varepsilon,a}$ we know that

$$\ln \rho_{\varepsilon,a_\varepsilon}(x) = -(2N+4) \ln |x| + o(\ln |x|) \quad \text{as } |x| \rightarrow \infty. \quad (3.7)$$

On the other hand, from the asymptotic formula (2.10)

$$\varepsilon^2 w_0(\varepsilon x) = -\frac{8\pi N(N+2)(1+\frac{\gamma}{2m})}{3(N+1)\gamma^2 \sin(\frac{\pi N}{N+1})} \varepsilon^2 \ln |x| + o(\ln |x|) \quad \text{as } |x| \rightarrow \infty. \quad (3.8)$$

Thus (1.16) holds true. Now, from (2.14) we obtain

$$|u_\varepsilon(x)| \leq C_1 \|u_\varepsilon\|_Y (\ln^+ |x| + 1) \leq C \|v_\varepsilon\|_{Y \times \mathbb{R}^2} (\ln^+ |x| + 1).$$

This implies

$$|u_\varepsilon(\varepsilon x)| \leq C_2 \|v_\varepsilon\|_{Y \times \mathbb{R}^2} (\ln^+ |\varepsilon x| + 1) \leq C \|v_\varepsilon\|_{Y \times \mathbb{R}^2} (\ln^+ |x| + 1). \quad (3.9)$$

From the continuity of the implicit function $\varepsilon \mapsto v_\varepsilon$ from $(-\varepsilon_0, \varepsilon_0)$ into $Y \times \mathbb{R}^2$ and the fact $v_0 = 0$ we have

$$\|v_\varepsilon\|_{Y \times \mathbb{R}^2} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (3.10)$$

Due to (3.10) the second inequality in (1.17) holds true. Let us check that the constructed solution to (1.5) and (1.6) is non-topological. This fact is the simple consequence of the asymptotic formula

$$\begin{aligned} u_\varepsilon(x) = & -(2N + 4 + \sigma\varepsilon^2 + o(\varepsilon^2)) \ln |x| \\ & + o(\ln |x|) \quad \text{as } |x| \rightarrow \infty, \quad \forall \varepsilon \in (0, \varepsilon_1), \end{aligned} \quad (3.11)$$

deduced from (3.7) to (3.10). In particular, (3.11) combined with (1.13) implies (1.18).

By (3.7)–(3.10) there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and a continuous function $\varepsilon \rightarrow \beta(\varepsilon) > 0$, $\lim_{\varepsilon \rightarrow 0} \beta(\varepsilon)/\varepsilon^2 = \sigma$ such that our solution $u_\varepsilon(x)$ of (1.5) and (1.6) satisfies

$$e^{u_\varepsilon(x)} = O\left(\frac{1}{|x|^{2N+4+\beta(\varepsilon)}}\right) \quad \text{as } |x| \rightarrow \infty. \quad (3.12)$$

We set $z = x_1 + ix_2$, $\bar{\partial}_z = \frac{1}{2}(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2})$ and define

$$A_{1,\varepsilon} = -\operatorname{Re}\{2i\bar{\partial}_z \ln \psi_\varepsilon(z)\}, \quad A_{2,\varepsilon} = -\operatorname{Im}\{2i\bar{\partial}_z \ln \psi_\varepsilon(z)\}.$$

Also recall that \mathcal{N}_ε is given by (2.29), ψ_ε is given by (1.13) and $A_{0,\varepsilon} = -\mathcal{N}_\varepsilon$. Then, $(\psi_\varepsilon, A_\varepsilon, \mathcal{N}_\varepsilon)$ becomes a solution to Eqs. (1.1)–(1.4) (see e.g. [12]). We now show that our solution $(\psi_\varepsilon, A_\varepsilon, \mathcal{N}_\varepsilon)$ is of finite energy, and satisfies the decay estimates (1.19). By (3.12)

$$|\psi_\varepsilon(x)|^2 = O\left(\frac{1}{|x|^{2N+4+\beta(\varepsilon)}}\right) \quad \text{as } |x| \rightarrow \infty. \quad (3.13)$$

Due to (3.12) and a priori estimate (2.16) applied to boundary-value problem (1.6) we have

$$\|\mathcal{N}_\varepsilon\|_{W^{2,2}(\mathbb{R}^2) \cap X} \leq C_3. \quad (3.14)$$

Therefore, by (1.6) and (3.14)

$$\mathcal{N}_\varepsilon^2 + |\nabla \mathcal{N}_\varepsilon|^2 + |\mathcal{N}_\varepsilon| |\psi_\varepsilon|^2 \in L^1(\mathbb{R}^2). \quad (3.15)$$

Next let us prove (1.19), which implies that the function

$$|D_1 \psi_\varepsilon(x)|^2 + |D_2 \psi_\varepsilon(x)|^2$$

belongs to $L^1(\mathbb{R}^2)$. From (1.2) we immediately have that $\bar{\partial}_z \ln \psi_\varepsilon(z) = -i\bar{\alpha}$, where $\bar{\alpha} = \frac{1}{2}(A_{1,\varepsilon} + iA_{2,\varepsilon})$, which, in turn, gives

$$\frac{1}{2} \frac{\partial \theta}{\partial x_1} - A_{1,\varepsilon} = -\frac{1}{2} \frac{\partial u_\varepsilon}{\partial x_2}, \quad \frac{1}{2} \frac{\partial \theta}{\partial x_2} - A_{2,\varepsilon} = \frac{1}{2} \frac{\partial u_\varepsilon}{\partial x_1}$$

and $D_1 \psi_\varepsilon = (\frac{1}{2} \frac{\partial u_\varepsilon}{\partial x_1} + i \frac{\partial \theta}{\partial x_1} - iA_{1,\varepsilon}) \psi_\varepsilon = \frac{1}{2} (\frac{\partial u_\varepsilon}{\partial x_1} - i \frac{\partial u_\varepsilon}{\partial x_2}) \psi_\varepsilon$ and $D_2 \psi_\varepsilon = (\frac{1}{2} \frac{\partial u_\varepsilon}{\partial x_2} + i \frac{\partial \theta}{\partial x_2} - iA_{2,\varepsilon}) \psi_\varepsilon = \frac{1}{2} (\frac{\partial u_\varepsilon}{\partial x_2} + i \frac{\partial u_\varepsilon}{\partial x_1}) \psi_\varepsilon$. Therefore,

$$|D_1 \psi_\varepsilon|^2 + |D_2 \psi_\varepsilon|^2 \leq \frac{1}{4} |\nabla u_\varepsilon|^2 |\psi_\varepsilon|^2 = \frac{1}{4} |\nabla u_\varepsilon|^2 e^{u_\varepsilon}.$$

By (1.5)

$$\begin{aligned} u_\varepsilon(x) = & -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |x-y| \left(\frac{q^2}{2m} e^{u_\varepsilon(y)} - 2\gamma \mathcal{N}_\varepsilon \right) dy \\ & + 2 \sum_{j=1}^k n_j \ln |x-z_j| + C_4 \end{aligned} \quad (3.16)$$

for some constant C_4 . Since by (3.11) the function $u_\varepsilon(x)$ is bounded from above for all $x \in \mathbb{R}^2$, taking derivative of u_ε from (3.16), and using the inequality $(\sum_{j=1}^n a_j)^2 \leq n \sum_{j=1}^n a_j^2$, we obtain

$$\begin{aligned} |\nabla u_\varepsilon(x)|^2 e^{u_\varepsilon(x)} \leq & \frac{q^4(k+2)}{8m^2\pi^2} e^{u_\varepsilon(x)} \left(\int_{\mathbb{R}^2} \frac{e^{u_\varepsilon(y)}}{|x-y|} dy \right)^2 \\ & + \frac{\gamma^2(k+2)}{\pi^2} e^{u_\varepsilon(x)} \left(\int_{\mathbb{R}^2} \frac{|\mathcal{N}_\varepsilon(y)|}{|x-y|} dy \right)^2 \\ & + 4(k+2) \sum_{j=1}^k n_j^2 \frac{e^{u_\varepsilon(x)}}{|x-z_j|^2} = I_1 + I_2 + I_3. \end{aligned}$$

Since $u_\varepsilon(x) = 2n_j \ln |x-z_j| + O(1)$ as $x \rightarrow z_j$, the function $I_3(x)$ is locally integrable and by (3.12)

$$I_3 = O\left(\frac{1}{|x|^{2N+4+\beta(\varepsilon)}}\right) \quad \text{as } |x| \rightarrow \infty.$$

Hence $I_3 \in L^1(\mathbb{R}^2)$.

From the estimate

$$\int_{\mathbb{R}^2} \frac{e^{u_\varepsilon(y)}}{|x-y|} dy \leq \int_{|x-y| \leq 1} \frac{e^{u_\varepsilon(y)}}{|x-y|} dy + \int_{|x-y| > 1} e^{u_\varepsilon(y)} dy \leq C_5,$$

where the constant C_5 is independent of $x \in \mathbb{R}^2$, and (3.12) we also find that

$$I_1 = O\left(\frac{1}{|x|^{2N+4+\beta(\varepsilon)}}\right) \quad \text{as } |x| \rightarrow \infty,$$

and is also integrable. Now we deduce the decay for I_2 .

Indeed, by (1.6) and (3.13)

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{|\mathcal{N}_\varepsilon(y)|}{|x-y|} dy &\leq \int_{|x-y| \leq 1} \frac{|\mathcal{N}_\varepsilon(y)|}{|x-y|} dy + \int_{|x-y| > 1} |\mathcal{N}_\varepsilon(y)| dy \\ &\leq C_6 \|\mathcal{N}_\varepsilon\|_{L^\infty(\mathbb{R}^2) \cap X} \leq C_7. \end{aligned}$$

This estimate implies that

$$I_2 = O\left(\frac{1}{|x|^{2N+4+\beta(\varepsilon)}}\right) \quad \text{as } |x| \rightarrow \infty.$$

The proof of (1.19) is complete. Finally, we note that since by (1.1) $F_{j0}^2 = \mathcal{N}_\varepsilon^2$ for $j = \{1, 2\}$ then by (3.13)–(3.15), (1.19) the energy $\mathcal{E}(A_\varepsilon, \psi_\varepsilon, \mathcal{N}_\varepsilon)$ is finite.

In order to prove (1.20) we set

$$G = \frac{q^2}{4\gamma m} e^{u_\varepsilon} - \mathcal{N}_\varepsilon.$$

Then, we have

$$\begin{aligned} \Delta G &= -\Delta \mathcal{N}_\varepsilon + \frac{\partial}{\partial \gamma \mu} \Delta v_\varepsilon \varepsilon^{v_\varepsilon} + |\nabla v_\varepsilon| \varepsilon^{v_\varepsilon} \\ &= -\gamma^2 \mathcal{N}_\varepsilon + \frac{q^2}{2} \left(1 + \frac{\gamma}{2m}\right) e^{u_\varepsilon} + \frac{q^2}{4\gamma m} e^{u_\varepsilon} \left(\frac{q^2}{2m} e^{u_\varepsilon} - 2\gamma \mathcal{N}_\varepsilon\right) + |\nabla u_\varepsilon|^2 e^{u_\varepsilon} \\ &= \left(\gamma^2 + \frac{q^2}{2m} e^{u_\varepsilon}\right) G + \frac{q^2}{2} e^{u_\varepsilon} + |\nabla u_\varepsilon|^2 e^{u_\varepsilon} \geq \left(\gamma^2 + \frac{q^2}{2m} e^{u_\varepsilon}\right) G \quad \text{in } \mathbb{R}^2. \end{aligned}$$

Also (1.7) yields zero boundary conditions at infinity

$$G(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

Thus, (1.20) follows immediately by the maximum principle. ■

4. ON THE NON-TOPOLOGICAL SOLUTIONS TO THE RELATIVISTIC MAXWELL–CHERN–SIMONS–HIGGS SYSTEM

The Lagrangian density for a relativistic self-dual Maxwell–Chern–Simons–Higgs system introduced in [14] is

$$\begin{aligned} \mathcal{L}(A, \phi, \mathcal{N}) = & -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{\gamma}{4} e^{\mu\nu\rho} F_{\mu\nu} A_\rho - D_\mu \phi \overline{D^\mu \phi} \\ & - \frac{1}{2} \partial_\mu \mathcal{N} \partial^\mu \mathcal{N} - q^2 \mathcal{N}^2 |\phi|^2 - \frac{1}{2} (q|\phi|^2 + \gamma \mathcal{N} - q)^2, \end{aligned}$$

where we are using the same notation as in the previous non-relativistic case, and $\phi = \phi_1 + i\phi_2$ is the complex Higgs field in this model. The static energy functional for this system is

$$\begin{aligned} \mathcal{E}(A, \phi, \mathcal{N}) = & \int_{\mathbb{R}^2} \left\{ \frac{1}{2} (F_{j0}^2 + F_{12}^2) + |D_0 \phi|^2 + |D_j \phi|^2 + \frac{1}{2} |\nabla \mathcal{N}|^2 \right. \\ & \left. + q^2 \mathcal{N}^2 |\phi|^2 + \frac{1}{2} (q|\phi|^2 + \gamma \mathcal{N} - q)^2 \right\} dx. \end{aligned}$$

In this model the self-duality equations coupled with the Gauss law constraint for the static fields (A, ϕ, \mathcal{N}) are

$$A_0 = -\mathcal{N}, \tag{4.1}$$

$$(D_1 + iD_2)\phi = 0 \quad \text{in } \mathbb{R}^2, \tag{4.2}$$

$$\partial_1 A_2 - \partial_2 A_1 + q|\phi|^2 + \gamma \mathcal{N} - q = 0 \quad \text{in } \mathbb{R}^2, \tag{4.3}$$

$$\Delta \mathcal{N} = \gamma q(|\phi|^2 - 1) + (\gamma^2 + 2q^2|\phi|^2)\mathcal{N} \quad \text{in } \mathbb{R}^2. \tag{4.4}$$

If (A, ϕ, \mathcal{N}) is a solution that makes $\mathcal{E}(A, \phi, \mathcal{N})$ finite, then the natural boundary conditions near infinity are either

$$|\phi|^2 \rightarrow 1 \quad \text{and} \quad \mathcal{N} \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \tag{4.5}$$

or

$$\phi \rightarrow 0 \quad \text{and} \quad \mathcal{N} \rightarrow \frac{q}{\gamma} \quad \text{as } |x| \rightarrow \infty. \quad (4.6)$$

The former is called topological, and the latter is called non-topological. For the topological boundary conditions in \mathbb{R}^2 , and for the periodic boundary conditions existence of multivortex solutions, asymptotic decay properties and the Abelian Higgs limit and the Chern–Simons limit problems are studied in [2,3]. In this paper we are considering the existence problem of finite energy solutions for the non-topological boundary conditions. We set as previously

$$\phi = \exp \left[\frac{u}{2} + i \sum_{j=1}^k n_j \arg(z - z_j) \right],$$

where $\{z_j\}_{j=1}^k$ and $\{n_j\}_{j=1}^k$ are the prescribed zeros of ϕ and their multiplicities, respectively. Then, by a reduction procedure similar to the previous case we obtain the equations

$$\Delta u = 2q^2(e^u - 1) + 2\gamma q \mathcal{N} + 4\pi \sum_{j=1}^k n_j \delta(z - z_j) \quad \text{in } \mathbb{R}^2, \quad (4.7)$$

$$\Delta \mathcal{N} = \gamma q(e^u - 1) + (\gamma^2 + 2q^2 e^u) \mathcal{N} \quad \text{in } \mathbb{R}^2, \quad (4.8)$$

$$u(x) \rightarrow -\infty, \quad \mathcal{N}(x) \rightarrow \frac{q}{\gamma} \quad \text{as } |x| \rightarrow \infty. \quad (4.9)$$

We set

$$\Psi_1(r) = \frac{1}{4\gamma q^3} \left(1 + \frac{2q^2}{\gamma^2} \right) \frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 - \frac{\gamma}{8q^5} \left(1 + \frac{2q^2}{\gamma^2} \right)^2 \rho^2. \quad (4.10)$$

Let $w_0(r)$ be a function defined by formulas (1.10) and (1.11) with $\tilde{f}(r) = -2\gamma q \Psi_1$.

For the non-topological multivortex solutions of the relativistic Maxwell–Chern–Simons–Higgs system our main result is:

THEOREM 4.1. *Let $\{z_j\}_{j=1}^k \subset \mathbb{C}^1$, $\{n_j\}_{j=1}^k \subset \mathbb{Z}_+$, $N = \sum_{j=1}^k n_j$, $n_j \geq 0$ and the function $\rho_{\varepsilon,a}$ be defined by (1.8). Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a solution $(\phi_\varepsilon, A_\varepsilon, \mathcal{N}_\varepsilon)$ to problem (4.1)–(4.4) with the following properties:*

1. *The energy functional $\mathcal{E}(\phi_\varepsilon, A_\varepsilon, \mathcal{N}_\varepsilon)$ is finite, and the function ϕ has the zeros $\{z_j\}_{j=1}^k$ with multiplicities $\{n_j\}_{j=1}^k$.*

2. The solutions $(\phi_\varepsilon, A_\varepsilon, \mathcal{N}_\varepsilon)$ could be represented by the formulas

$$\phi_\varepsilon(z) = e^{\left[\frac{u_\varepsilon}{2} + i \sum_{j=1}^k n_j \arg(z - z_j)\right]}, \quad (4.11)$$

$$u_\varepsilon(z) = \ln\left(\frac{\gamma}{4q^2} \rho_{\varepsilon, a_\varepsilon}(z)\right) + \varepsilon^2 w_0(\varepsilon|z|) + \varepsilon^2 u_\varepsilon(\varepsilon z), \quad (4.12)$$

$$\mathcal{N}_\varepsilon(z) = -\frac{q}{\gamma} \left(1 + \frac{2q^2}{\gamma^2}\right) e^{u_\varepsilon(z)} + \frac{q}{\gamma} + \Psi_\varepsilon(z) + \varepsilon^4 \Psi_1(\varepsilon z), \quad (4.13)$$

where $a_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, and

$$w_0(\varepsilon|z|) = -\sigma_1 \ln|z| + o(\ln|z|) \quad \text{as } |z| \rightarrow +\infty, \quad (4.14)$$

where

$$\sigma_1 = \frac{2\pi N(N+2)(1 + \frac{2q^2}{\gamma^2})^2}{3(N+1)q^4 \sin(\frac{\pi N}{N+1})},$$

$$\|\Psi_\varepsilon\|_{L^2(\mathbb{R}^2)} = o(\varepsilon^3), \quad \|u_\varepsilon / \ln(|z| + 1)\|_{C^0(\mathbb{R}^2)} \leq o(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow +0. \quad (4.15)$$

3. There exists a constant $\hat{C} = \hat{C}(\gamma, q)$ and a function $\beta(\varepsilon)$ such that solutions $(\phi_\varepsilon, A_\varepsilon, \mathcal{N}_\varepsilon)$ satisfy the decay estimate,

$$\ln|\phi_\varepsilon(z)|^2 = -(2N+4 + \sigma_1 \varepsilon^2 + o(\varepsilon^2)) \ln|z| + o(\ln|z|) \quad \text{as } |z| \rightarrow \infty, \quad (4.16)$$

$$|D_1 \phi_\varepsilon(z)|^2 + |D_2 \phi_\varepsilon(z)|^2 \leq \frac{\hat{C}}{|z|^{2N+4+\beta(\varepsilon)}} + o\left(\frac{1}{|z|^{2N+4+\beta(\varepsilon)}}\right) \quad \text{as } |z| \rightarrow \infty, \quad (4.17)$$

where $\beta(\varepsilon) > 0$ for all $\varepsilon \in (0, \varepsilon_0)$ and $\lim_{\varepsilon \rightarrow +0} \beta(\varepsilon)/\varepsilon^2 = \sigma_1$.

Proof. If we transform the unknowns, $(u, \mathcal{N}) \rightarrow (u, S)$ by

$$\mathcal{N} = -S - \left(\frac{q}{\gamma} + \frac{2q^3}{\gamma^3}\right) e^u + \frac{q}{\gamma}, \quad (4.18)$$

then by elementary computation we find that (4.7)–(4.9) is reduced to

$$\Delta u = -2\gamma q S - \frac{4q^4}{\gamma^2} e^u + 4\pi \sum_{j=1}^k n_j \delta(z - z_j) \quad \text{in } \mathbb{R}^2, \quad (4.19)$$

$$\begin{aligned} \Delta S = & \gamma^2 S + \left(\frac{2q^3}{\gamma} + \frac{8q^5}{\gamma^3} + \frac{8q^7}{\gamma^5} \right) e^{2u} + \left(4q^2 + \frac{4q^4}{\gamma^2} \right) S e^u \\ & - \left(\frac{q}{\gamma} + \frac{2q^3}{\gamma^3} \right) |\nabla u|^2 e^u \quad \text{in } \mathbb{R}^2, \end{aligned} \quad (4.20)$$

$$u(x) \rightarrow -\infty, \quad S(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (4.21)$$

We find that system (4.19)–(4.21) is the same as (2.2)–(2.4) except for changes in constant coefficients. Thus we can apply exactly the same argument as in the proof of Theorem 1.1. So we will be brief in the proof below, pointing out the steps which need care. We make a sequence of change of variables as before to get the functional:

$$\begin{aligned} P(u, a, \varepsilon) = & \Delta u + \frac{4q^4}{\gamma^2} g_{\varepsilon, a} \frac{e^{\varepsilon^2 u + \varepsilon^2 w_0} - 1}{\varepsilon^2} + 2\gamma q G_1(u, a, \varepsilon, G_2(u, a, \varepsilon)) \\ & + \Delta w_0 + 2\gamma q \Psi_1, \end{aligned}$$

where in this case

$$g_{\varepsilon, a}(z) = \frac{2\gamma^2 \varepsilon^{2N} |f(\frac{z}{\varepsilon})|^2}{q^4 (1 + \varepsilon^{2N+2} |F(\frac{z}{\varepsilon}) + \frac{a}{\varepsilon^{N+1}}|^2)^2}$$

with the same complex functions $f(z)$ and $F(z)$ are defined in (1.9). The function $w_0(r) \in Y$, introduced earlier is a solution of the ordinary differential equation

$$L_1 w_0 = \frac{1}{r} \frac{d}{dr} \left(r \frac{dw_0}{dr} \right) + \rho w_0 = -2\gamma q \Psi_1 \quad \text{in } \mathbb{R}_+, \quad (4.22)$$

where the function $\rho(r)$ is defined by (1.8). Similar to the previous case the mapping G_1 is defined as follows: $G_1(u, a, \varepsilon, f) = \Phi$ is a solution of the problem

$$\begin{aligned} \varepsilon^2 \Delta \Phi - \gamma^2 \Phi - 4\varepsilon^2 q^2 \left(1 + \frac{q^2}{\gamma^2} \right) g_{\varepsilon, a} e^{\varepsilon^2 u + \varepsilon^2 w_0} \Phi &= f \quad \text{in } \mathbb{R}^2, \\ \Phi(x) &\rightarrow 0 \quad \text{as } |x| \rightarrow \infty \end{aligned}$$

with (u, a, ε, f) belonging to the same function spaces as before. And $G_2 \times (u, a, \varepsilon)$ is the mapping defined by

$$\begin{aligned}
G_2(u, a, \varepsilon) = & -\frac{q}{\gamma} \left(1 + \frac{2q^2}{\gamma^2}\right) |\nabla(\varepsilon^2 u + \varepsilon^2 w_0 + \ln g_{\varepsilon, a})|^2 e^{\varepsilon^2 u + \varepsilon^2 w_0} g_{\varepsilon, a} \\
& + \frac{2q^3}{\gamma} \left(1 + \frac{2q^2}{\gamma^2}\right)^2 e^{2(\varepsilon^2 u + \varepsilon^2 w_0)} g_{\varepsilon, a}^2 + 4\varepsilon^2 q^2 \left(1 + \frac{q^2}{\gamma^2}\right) \\
& e^{\varepsilon^2 u + \varepsilon^2 w_0} g_{\varepsilon, a} \Psi_1 + \gamma^2 \Psi_1 - \varepsilon^2 \Delta \Psi_1.
\end{aligned}$$

Let $\varepsilon \mapsto (u_\varepsilon, a_\varepsilon)$ be an implicit function satisfying

$$P(u_\varepsilon, a_\varepsilon, \varepsilon) = 0,$$

then the pair

$$u_\varepsilon(x) = \ln \left(\frac{\gamma}{4q^2} \rho_{\varepsilon, a_\varepsilon}(x) \right) + \varepsilon^2 w_0(\varepsilon x) + \varepsilon^2 u_\varepsilon(\varepsilon x),$$

$$\mathcal{N}_\varepsilon(x) = -\frac{q}{\gamma} \left(1 + \frac{2q^2}{\gamma^2}\right) e^{u_\varepsilon(x)} + \frac{q}{\gamma} + \varepsilon^4 \Psi(\varepsilon x) + \varepsilon^4 \Psi_1(\varepsilon x)$$

with

$$\Psi(x) = G_1(u_\varepsilon, a_\varepsilon, \varepsilon, G_2(u_\varepsilon, a_\varepsilon, \varepsilon)),$$

is the solution to Eqs. (4.7) and (4.8). The two parts where the proof is not straightforwardly the same as before are the following:

LEMMA 4.1. *The function $w_0(r) \in Y$ solves Eq. (4.22) and satisfies the pointwise estimate:*

$$|w_0(r)| \leq C(\ln^+ r + 1), \quad \forall r > 0.$$

Moreover we have the asymptotic formula

$$w_0(r) = -\frac{2\pi N(N+2)(1 + \frac{2q^2}{\gamma^2})^2}{3(N+1)q^4 \sin(\frac{\pi N}{N+1})} \ln r + o(\ln r) \quad \text{as } r \rightarrow \infty.$$

PROPOSITION 4.1. *The following inequality holds true:*

$$\begin{aligned}
\tilde{C}_\pm := & \int_{\mathbb{R}^2} \left[\left\{ 4\rho w_0 + \frac{2}{q^2} \left(1 + \frac{2q^2}{\gamma^2}\right) \left(\frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 - \frac{\gamma^2}{q^2} \left(1 + \frac{2q^2}{\gamma^2}\right) \rho^2 \right) \right\} \varphi_\pm^2 \right. \\
& \left. + \frac{4}{q^2} \left(1 + \frac{2q^2}{\gamma^2}\right) \frac{d\rho}{dr} \frac{\partial \varphi_\pm}{\partial r} \varphi_\pm \right] dx < 0.
\end{aligned}$$

The proofs are given in the next section. The remaining parts of the proofs of the existence and the asymptotic behaviors of the non-topological solutions are the same as in the previous sections.

5. PROOF OF AUXILIARY LEMMAS

In this section we prove Lemmas 2.1 and 4.1, and Propositions 3.4 and 4.1. For the proof of these we first recall the following formula for the Mellin transform (see e.g. [13]).

$$\int_0^\infty \frac{t^{a-1}}{(1+t)^n} dt = \frac{\pi |(a-1)(a-2)\cdots(a-(n-1))|}{(n-1)!\sin(\pi a)}, \quad a \in (0, 1). \quad (5.1)$$

Proof of Lemma 2.1. We recall that given $\tilde{f}(|x|) \in C^1(\mathbb{R}_+) \cap X$, the ordinary differential equation (2.8) has a solution $w(r) \in Y$ given by the formulas (1.10) and (1.11) [4], where $\phi_{\tilde{f}}(1)$ and $w(1)$ are defined as limits of $\phi_{\tilde{f}}(r)$ and $w(r)$ as $r \rightarrow 1$.

From this explicit solution formula we find that $w_0(\cdot) \in C^0(\mathbb{R}_+)$, and thus (2.9) follows from (2.14). It suffices now to prove (2.10). We observe from formula (1.10) that

$$w_0(r) = \varphi_0(r) \int_2^r \left(\frac{1+s^{2N+2}}{1-s^{2N+2}} \right)^2 \frac{I(s)}{s} ds + (\text{bounded function of } r)$$

as $r \rightarrow \infty$, where

$$I(s) = \int_0^s \varphi_0(t) t \tilde{f}(t) dt.$$

Since $\varphi_0(r) \rightarrow -1$ as $r \rightarrow \infty$, (2.10) follows if we show

$$I = I(\infty) = \int_0^\infty \varphi_0(r) r \tilde{f}(r) dr = \frac{8\pi N(N+2)(1+\frac{\gamma}{2m})}{3(N+1)\gamma^2 \sin(\frac{\pi N}{N+1})}$$

for

$$\tilde{f}(r) = -2\gamma \Psi_0 = \frac{1}{\lambda} \left[\rho^2 - \frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 \right] = \frac{1}{\lambda} f_1(r) + \frac{1}{\lambda} f_2(r),$$

where we set $\lambda = \frac{\gamma a}{2b}$,

$$f_1(r) = \rho^2, \quad f_2(r) = -\frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2.$$

Then, we write

$$I = I_1 + I_2 = \frac{1}{\lambda} \int_0^\infty \varphi_0(r) r f_1(r) dr + \frac{1}{\lambda} \int_0^\infty \varphi_0(r) r f_2(r) dr.$$

We first compute explicitly to get $f_1(r) = \frac{64(N+1)^4 r^{4N}}{(1+r^{2N+2})^4}$. Thus, substituting $r^{2N+2} = t, \frac{N}{N+1} = a$ and integrating by parts we have

$$\begin{aligned} \lambda I_1 &= \int_0^\infty \varphi_0(r) r f_1(r) dr = 32(N+1)^3 \int_0^\infty \frac{t^{a-1}(1-t)t}{(1+t)^5} dt \\ &= 32(N+1)^3 \int_0^\infty t^{a-1} \left(\frac{a}{4(1+t)^4} - \frac{a(a+1)}{12(1+t)^3} \right) dt \\ &= 32(N+1)^3 J. \end{aligned} \tag{5.2}$$

We use formula (5.1), and find

$$\begin{aligned} J &= -\frac{\pi}{4! \sin \pi a} \{a(a-1)(a-2)(a-3) - a(a+1)(a-1)(a-2)\} \\ &= -\frac{\pi a(a-1)^2(a-2)}{12 \sin \pi a} = \frac{\pi N(N+2)}{12(N+1)^4 \sin(\frac{\pi N}{N+1})}. \end{aligned}$$

Thus, we finally obtain

$$\lambda I_1 = \frac{8\pi N(N+2)}{3(N+1) \sin(\frac{\pi N}{N+1})}. \tag{5.3}$$

We now claim $I_2 = 0$. Taking the scalar product in $L^2(\mathbb{R}_+)$ of the equation

$$\frac{1}{r\rho} \frac{d\rho}{dr} + \frac{1}{\rho} \frac{d^2\rho}{dr^2} - \frac{1}{\rho^2} \left(\frac{d\rho}{dr} \right)^2 + \rho = 0 \left[\Leftrightarrow \frac{1}{r} \frac{d}{dr} \left(r \frac{d\rho}{dr} \right) + \rho = 0 \right], \tag{5.4}$$

with the function $r\rho\varphi_0$ and integrating by parts we obtain

$$\begin{aligned} -\lambda I_2 &= \int_0^\infty \frac{r}{\rho} \left(\frac{d\rho}{dr} \right)^2 \varphi_0 dr \\ &= \int_0^\infty \frac{d\rho}{dr} \varphi_0 dr + \int_0^\infty \frac{d^2\rho}{dr^2} \varphi_0 r dr + \int_0^\infty \rho^2 \varphi_0 r dr \\ &= \int_0^\infty \rho r \left(\frac{d^2\varphi_0}{dr^2} + \frac{1}{r} \frac{d\varphi_0}{dr} \right) dr + \int_0^\infty \rho^2 \varphi_0 r dr \\ &= \int_0^\infty \rho r \Delta \varphi_0 dr + \int_0^\infty \rho^2 \varphi_0 r dr \\ &= - \int_0^\infty \rho^2 \varphi_0 r dr + \int_0^\infty \rho^2 \varphi_0 r dr = 0, \end{aligned} \tag{5.5}$$

where we used the fact $\varphi_0 \in \ker(\Delta + \rho)$ (see [4]). Thus,

$$I = \frac{1}{\tilde{\lambda}} I_1 = \frac{16\pi b N(N+2)}{3\gamma d(N+1) \sin(\frac{\pi N}{N+1})}.$$

This completes the proof of the lemma. ■

Proof of Proposition 3.4. Since $\tilde{C}_+ = \tilde{C}_-$ it suffices to prove $\tilde{C}_+ < 0$. In order to simplify the computations first let us transform some terms in \tilde{C}_+ :

$$\begin{aligned} & \int_{\mathbb{R}^2} \left(2 \frac{d\rho}{dr} \frac{\partial \varphi_+}{\partial r} \varphi_+ + \frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 \varphi_+^2 \right) dx \\ &= -\pi \int_0^\infty \frac{d}{dr} \left(\frac{r}{\rho} \frac{d\rho}{dr} \right) \rho \frac{r^{2N+2}}{(1+r^{2N+2})^2} dr \\ &= \pi \int_0^\infty \rho^2 \frac{r^{2N+3}}{(1+r^{2N+2})^2} dr. \end{aligned} \tag{5.6}$$

Using (5.6) and the identity

$$L_1 \left[\frac{1}{16(1+r^{2N+2})^2} \right] = \frac{(N+1)^2 r^{4N+2}}{(1+r^{2N+2})^4},$$

after integration by parts we obtain

$$\begin{aligned} \frac{\lambda}{4} \tilde{C}_+ &= \int_0^\infty r \left\{ \lambda w_0 L_1 \left[\frac{1}{2(1+r^{2N+2})^2} \right] - 2\rho^2 \phi^2 + \rho^2 \phi^2 \right\} dr \\ &\quad \times \int_0^{2\pi} \cos^2(N+1)\theta d\theta \\ &= \pi \int_0^\infty \left\{ \lambda (L_1 w_0) \frac{1}{2(1+r^{2N+2})^2} - 2\rho^2 \phi^2 + \rho^2 \phi^2 \right\} r dr \\ &= \pi \int_0^\infty r \left\{ \left(\rho^2 - \frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 \right) \frac{1}{2(1+r^{2N+2})^2} - 2\rho^2 \phi^2 + \rho^2 \phi^2 \right\} dr \\ &= \pi \int_0^\infty r \left\{ \frac{\rho^2}{2(1+r^{2N+2})^2} - 2\rho^2 \phi^2 \right\} dr \\ &\quad - \pi \int_0^\infty r \left\{ \frac{1}{2\rho} \left(\frac{d\rho}{dr} \right)^2 \frac{1}{(1+r^{2N+2})^2} - \rho^2 \phi^2 \right\} dr := J_1 - J_2. \end{aligned}$$

Clearly,

$$J_1 = 64\pi(N+1)^4 \int_0^\infty \left\{ \frac{5r^{4N+1}}{2(1+r^{2N+2})^6} - \frac{2r^{4N+1}}{(2+r^{2N+2})^5} \right\} dr.$$

Substituting $r^{2N+2} = t$, $\frac{N}{N+1} = a$ in the previous equality we have the following:

$$\begin{aligned}
 J_1 &= 32\pi(N+1)^3 \int_0^\infty \left\{ \frac{5t^a}{2(1+t)^6} - \frac{2t^a}{(1+t)^5} \right\} dt \\
 &= -32\pi(N+1)^3 \int_0^\infty \left\{ \frac{t^a}{2} \frac{d}{dt} \left[\frac{1}{(1+t)^5} \right] - \frac{1}{2} t^a \frac{d}{dt} \left[\frac{1}{(1+t)^4} \right] \right\} dt \\
 &= 16\pi(N+1)^2 N \int_0^\infty \left\{ \frac{t^{a-1}}{(1+t)^5} - \frac{t^{a-1}}{(1+t)^4} \right\} dt \\
 &= \frac{16\pi^2(N+1)^2 N}{\sin(\pi a)} \left\{ \frac{(a-1)(a-2)(a-3)(a-4)}{4!} - \frac{|(a-1)(a-2)(a-3)|}{3!} \right\} \\
 &= \frac{2\pi^2 N(a-1)(a-2)}{3 \sin(\pi a)} \{(2N+3)(3N+4) - 4(N+1)(2N+3)\} \\
 &= -\frac{2\pi^2 N^2(a-1)(a-2)(2N+3)}{3 \sin(\pi a)} < 0, \quad \forall N \geq 1.
 \end{aligned}$$

In order to transform the second term J_2 we take the scalar product in $L^2(\mathbb{R}_+)$ of Eq. (5.4) and the function $\frac{r\rho}{2(1+r^{2N+2})^2}$. Then after integration by parts we obtain

$$\begin{aligned}
 &\int_0^\infty \left(\frac{\rho^2 r}{(1+r^{2N+2})^2} - \left(\frac{d\rho}{dr} \right)^2 \frac{r}{\rho(1+r^{2N+2})^2} \right) dr \\
 &= \int_0^\infty r \frac{d\rho}{dr} \frac{d}{dr} \left(\frac{1}{(1+r^{2N+2})^2} \right) dr \\
 &= -(4N+4) \int_0^\infty \frac{d\rho}{dr} \left(\frac{1}{(1+r^{2N+2})^2} - \frac{1}{(1+r^{2N+2})^3} \right) dr \\
 &= - \int_0^\infty \left(\frac{2\rho^2 r}{(1+r^{2N+2})} - \frac{3\rho^2 r}{(1+r^{2N+2})^2} \right) dr.
 \end{aligned}$$

Therefore

$$\begin{aligned}
& \int_0^\infty \left(\frac{d\rho}{dr} \right)^2 \frac{r}{\rho(1+r^{2N+2})^2} dr \\
&= \int_0^\infty \left(\frac{2\rho^2 r}{(1+r^{2N+2})} - \frac{2\rho^2 r}{(1+r^{2N+2})^2} \right) dr = 2 \int_0^\infty \rho^2 \phi^2 r dr.
\end{aligned}$$

Hence $J_2 = 0$ and

$$\tilde{C}_+ = \frac{4}{\lambda} J_1 < 0$$

for $N \geq 1$. This proves the lemma. ■

Proof of Lemma 4.1. Let us set

$$I(s) = \int_0^s \varphi_0(t) t \tilde{f}(t) dt,$$

where

$$\begin{aligned}
\tilde{f}(r) &= \frac{\gamma^2}{4q^4} \left(1 + \frac{2q^2}{\gamma^2} \right)^2 \rho^2 - \frac{1}{2q^2} \left(1 + \frac{2q^2}{\gamma^2} \right) \frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 \\
&:= \frac{\gamma^2}{4q^4} \left(1 + \frac{2q^2}{\gamma^2} \right)^2 f_1(r) + \frac{1}{2q^2} \left(1 + \frac{2q^2}{\gamma^2} \right) f_2(r)
\end{aligned}$$

with the same $f_1(r)$ and $f_2(r)$ as in the proof of Lemma 2.1. Then, as in the proof of Lemma 2.1, since $\varphi_0(r) \rightarrow -1$ as $r \rightarrow \infty$, Lemma 4.1 follows if we show that

$$I = I(\infty) = \frac{2\pi N(N+2)(1 + \frac{2q^2}{\gamma^2})^2}{3(N+1)q^4 \sin(\frac{\pi N}{N+1})}.$$

We have the decomposition

$$\begin{aligned}
I &= \frac{\gamma^2}{4q^4} \left(1 + \frac{2q^2}{\gamma^2} \right)^2 \int_0^\infty \varphi_0(r) r \tilde{f}(r) dr \\
&= \int_0^\infty \varphi_0(r) r f_1(r) dr + \frac{1}{2q^2} \left(1 + \frac{2q^2}{\gamma^2} \right) \int_0^\infty \varphi_0(r) r f_2(r) dr := I_1 + I_2.
\end{aligned}$$

By (5.5) $I_2 = 0$, and by (5.3)

$$\int_0^\infty \varphi_0(r) r f_1(r) dr = \frac{8\pi N(N+2)}{3(N+1)\sin(\frac{\pi N}{N+1})}.$$

Therefore

$$I = I_1 = \frac{2\pi\gamma^2(1 + \frac{2q^2}{\gamma^2})^2 N(N+2)}{3q^4(N+1)\sin(\frac{\pi N}{N+1})}.$$

The proof of the lemma is complete. ■

Proof of Proposition 4.1. As in the proof of Proposition 3.4 we transform \tilde{C}_\pm into

$$\begin{aligned} \tilde{C}_\pm &= \pi \int_0^\infty \left\{ 4(L_1 w_0) \frac{1}{2(1+r^{2N+2})^2} + \left[\frac{2}{q^2} \left(1 + \frac{2q^2}{\gamma^2} \right) \left\{ \frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 \right. \right. \right. \\ &\quad \left. \left. - 2 \frac{\gamma^2}{q^2} \left(1 + \frac{2q^2}{\gamma^2} \right) \rho^2 - \frac{d^2 \rho}{dr^2} \right\} - \frac{1}{r} \frac{d\rho}{dr} \right] \phi^2 \right\} r dr \\ &\quad \left(\text{using } L_1 w_0 = \frac{\gamma^2}{4q^4} \left(1 + \frac{2q^2}{\gamma^2} \right)^2 \rho^2 - \frac{1}{2q^2} \left(1 + \frac{2q^2}{\gamma^2} \right) \frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 \right) \\ &= \frac{\pi\gamma^2}{q^4} \left(1 + \frac{2q^2}{\gamma^2} \right)^2 \int_0^\infty r \left[\frac{\rho^2}{2(1+r^{2N+2})^2} - 2\rho^2 \phi^2 \right] dr \\ &\quad - \frac{2\pi}{q^2} \left(1 + \frac{2q^2}{\gamma^2} \right) \int_0^\infty r \left[\frac{1}{2\rho(1+r^{2N+2})^2} \left(\frac{d\rho}{dr} \right)^2 \right. \\ &\quad \left. - \frac{1}{\rho} \left(\frac{d\rho}{dr} \right)^2 \phi^2 + \frac{d^2 \rho}{dr^2} \phi^2 + \frac{1}{r} \frac{d\rho}{dr} \phi^2 \right] dr \\ &:= \frac{\pi\gamma^2}{q^4} \left(1 + \frac{2q^2}{\gamma^2} \right)^2 J_1 - \frac{2\pi}{q^2} \left(1 + \frac{2q^2}{\gamma^2} \right) J_2. \end{aligned}$$

In the proof of Proposition 3.4 we showed $J_1 < 0$, and $J_2 = 0$. Thus, $\tilde{C}_\pm < 0$. This completes proof of the proposition. ■

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